MATHEMATICS

ISOGENIES OF FORMAL GROUPS

BY

FRANS OORT

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Let k be a field of characteristic p, let G be a formal group (or a p-divisable group, or a commutative group scheme) over k. We define

$$a(G) := \dim_{\bar{k}} \operatorname{Hom}(\alpha_p, G \otimes \bar{k}),$$

and we like to know which values $a(G_1)$ can have for a formal group $G_1 \sim G$ (i.e. G_1 isogenous with G). We show that $a(G_1) \leqq 1$ can be realized over $\bar{k}$, or by taking generic quotients. This last result is one of the technical details used by Mumford in his proof (unpublished) of liftability of abelian varieties to characteristic zero.

SOME NOTATIONS

We consider commutative group schemes, and all schemes will be over a field of characteristic p. The proofs are concerned with finite group schemes and formal groups of local-local type (in the sense of [3], I.2). If G is a formal group over an algebraically closed field, we denote by $A(G)$ or by AG the subgroup scheme of G generated by all images $\alpha_p \to G$; thus, if $a(G)=a$, then $AG \cong (\alpha_p)^a$. We shall use freely notations and results from the basic paper [2] by MANIN. We denote by $\mathcal{N}_{p,k}$ the category of finite commutative k-group schemes of rank power of p, and by Ind $(\mathcal{N}_{p,k})$ we denote the category of inductive systems in it. We write BT group (Barsotti-Tate group) instead of p-divisable group.

1. *The minimum of* $a(-)$

THEOREM (1.1). Let F be a BT group over a field k. Then there exists an isogeny $F \to F'$ over a finite extension of k such that $a(F') \leqslant 1$.

For the proof we may assume k to be algebraically closed, and F (and F') purely of local-local type. Up to isogeny over $k=\bar{k}$ any BT group is a direct sum of isosimple groups, and we assume F to be isomorphic over k with a direct sum of isosimple groups. These summands all can be taken of the form $G_{n,m}$, $n>0$, $m>0$, $(n,m)=1$ (in the notation of [2], II.4, page 35).

LEMMA (1.2). All endomorphisms of $G_{n,m}$ are defined over the finite field $\mathbf{F}(p^{n+m})$. Let G and H be BT groups over a field k; the group $\mathrm{Hom}_k(H, G)$ is a finitely generated $\mathbf{Z}_p$-module.

PROOF: The first statement follows from [2], Lemma 3.7 on page 46. Over an algebraic closure of k one can choose isogenies

$$\oplus H_i \to H \text{ and } G \to \oplus G_j$$

where H_i and G_j are of type $G_{n,m}$. This yields an injective homomorphism

$$\mathrm{Hom}_k(H, G) \to \mathrm{Hom}_k(\oplus H_i, \oplus G_j).$$

Let $W = W_\infty(\mathbf{F}(p^{n+m}))$, and $\mathscr{A} := W[[F, V]]$ the ring used for the Dieudonné-module theory. Then the Dieudonné module of $G_{n,m}$ is

$$M_{n,m} = \mathscr{A}/\mathscr{A}(F^m - V^n)$$

(cf. [2], p. 35), and by the first result of this lemma we conclude

$$\mathrm{End}_k(G_{n,m}) = \mathrm{End}_{\mathscr{A}}(M_{n,m}) \cong M_{n,m}.$$

Thus $\mathrm{Hom}_k(H_i, G_j)$ is finitely generated over $\mathbf{Z}_p$, hence the same holds for $\mathrm{Hom}_k(H, G)$, which proves the lemma.

LEMMA (1.3). Let G and H be BT groups over $k = \bar{k}$. Assume $a(G) = 1 = a(H)$, and suppose H is isosimple. Then there exists a finite k-subgroup scheme $N \subset G \oplus H$ such that

$$a(G \oplus H/N) = 1.$$

PROOF: Choose isomorphisms

$$f: \alpha_p \xrightarrow{\sim} A(G) \subset G, \quad g: \alpha_p \xrightarrow{\sim} A(H) \subset H.$$

Then we obtain

$$\Phi: \mathrm{Hom}_k(H, G) \to \mathrm{Hom}_k(\alpha_p, \alpha_p) \cong k,$$

$$\Phi(\varphi) := f^{-1} \cdot \varphi \cdot g.$$

From the previous lemma we deduce that the image of Φ is *finite*; we fix

$$(i_1, i_2) = \xi: \alpha_p \to G \oplus H, \quad \frac{i_1}{i_2} \notin \mathrm{Im}\, \Phi$$

(i.e. there does not exist $\varphi \in \mathrm{Hom}_k(H, G)$ such that $\Phi(\varphi) = f^{-1} i_1 i_2^{-1} g$). Assume the lemma is false, i.e. for any finite $M \subset G \oplus H$,

$$a(G \oplus H/M) \geqslant 2.$$

We then construct inductively $\{N_i\}_{i=1}^{\infty}$, with $N_i \subset N_{i+1} \subset G \oplus H$ such that $a(N_i) = 1$ for all i, and

$$\mathrm{Im}\,(\alpha_p = A(N_i) \hookrightarrow N_i \hookrightarrow G \oplus H) = \mathrm{Im}\,(\xi: \alpha_p \to G \oplus H).$$

This we achieve as follows: take $N_1 := \xi(\alpha_p)$; assume N_i constructed, $i \geqslant 1$, and construct N_{i+1} as follows:

$$\begin{tikzcd}
 & & N_{i+1} \arrow[r] \arrow[d, hook] & \alpha_p \arrow[d, hook, "h"] \arrow[dl, dashed, "\nexists"'] & \\
0 \arrow[r] & N_i \arrow[r] & G \oplus H \arrow[r, "\pi"'] & G \oplus H/N_i \arrow[r] & 0;
\end{tikzcd}$$

by assumption $a(G \oplus H/N_i) \geqslant 2$; note that $a(G \oplus H) = 2$, and $a(N_i) = 1$, thus

$$\operatorname{Hom}(\alpha_p, G \oplus H) \to \operatorname{Hom}(\alpha_p, G \oplus H/N_i)$$

is not surjective; choose $h \in \operatorname{Hom}(\alpha_p, G \oplus H/N_i)$ outside the image and construct

$$N_{i+1} := \overset{-1}{\pi}(h(\alpha_p)).$$

If $a(N_{i+1})$ would be bigger than one, there would exist $\alpha_p \to N_{i+1}$ not factoring through $N_i \subset N_{i+1}$, a contradiction with the fact that

$$h: \alpha_p \to G \oplus H/N_i$$

does not lift to $G \oplus H$; hence $a(N_{i+1}) = 1$; clearly $A(N_i) = A(N_{i+1})$, and the sequence $\{N_i\}_{i=1}^{\infty}$ with the required properties is constructed. Because $(i_1: \alpha_p \to G) \neq 0$, we know $N_i \cap G = 0$, thus p_i is injective:

$$\begin{tikzcd}
N_i \arrow[r, hook] \arrow[dr, hook, "p_i"'] & G \oplus H \arrow[d, "\text{proj.}"] \\
 & H.
\end{tikzcd}$$

Let

$$Q := \bigcup_{i=1}^{\infty} N_i \subset G \oplus H, \ p: Q \to H$$

(union taken in $\operatorname{Ind}(\mathscr{N}_p)$). We conclude that $p: Q \xrightarrow{\sim} H$: by the classification theorem the object Q is isogenous with a direct sum of isosimple factors, at least one of these factors must be non-zero, because Q is not finite, thus there exists a non-zero BT group Q' and an isogeny

$$Q' \to Q \subset H;$$

because H is isosimple this implies $\operatorname{Im}(Q' \to H) = H$, hence $Q \cong H$.

Thus Q is the graph of a morphism:

$$\varphi := (H \xrightarrow{p^{-1}} Q \hookrightarrow G \oplus H \to G)$$

and

$$\Phi(\varphi) = \frac{i_1}{i_2},$$

a contradiction; this proves the lemma.

Remark (1.4). T. Oda suggested to me the following alternative proof of (1.3). Assume $H = G_{n,m}$. Let M, $M_{n,m}$, respectively k be the Dieudonné modules of G, H, respectively α_p. We choose $\xi \in \operatorname{Ext}^1_{\mathscr{A}}(M, M_{n,m})$,

$$(\xi) \quad 0 \to M_{n,m} \to N \to M \to 0$$

such that if

$$a\colon M_{n,m} \twoheadrightarrow k, \text{ then } 0 \neq a_*(\xi) \in \operatorname{Ext}^1_{\mathscr{A}}(M, k).$$

This is possible, because a resolution

$$0 \to \mathscr{A} \to \mathscr{A} \to M \to 0$$

yields

$$\begin{tikzcd}
\operatorname{Hom}(\mathscr{A}, M_{n,m}) \arrow[r] \arrow[d] & \operatorname{Ext}^1(M, M_{n,m}) \arrow[d, "a_*"] \\
0 \neq \operatorname{Hom}(\mathscr{A}, k) \arrow[r, hook] & \operatorname{Ext}^1(M, k).
\end{tikzcd}$$

The extension ξ yields an exact sequence

$$0 \to \operatorname{Hom}(M, k) \xrightarrow{\sim} \operatorname{Hom}(N, k) \to \operatorname{Hom}(M_{n,m}, k) \xrightarrow{\delta} \operatorname{Ext}^1(M, k);$$

because

$$\dim_k(\operatorname{Hom}(M_{n,m}, k)) = 1,$$

and $\delta \neq 0$, we conclude δ to be injective, and hence $\operatorname{Hom}(N, k) = k$.

Corollary (1.5). Let k be an algebraically closed field, and G a BT group over k. Assume G is isogenous with

$$H = \bigoplus_i G_{n_i, m_i};$$

we write

$$n := \sum \min(n_i, m_i).$$

Then

$$a(G) \leqslant n.$$

Suppose conversely given H, and an integer m with $1 \leqslant m \leqslant n$; then there exists a BT group G isogenous with H such that $a(G) = m$.

PROOF. If H_1 is a p-divisable group, we write

$$\max a(H_1) := \max (a(G)),$$

the maximum taken over all G isogenous with H_1. First we note that

$$\max a(H_1 \oplus H_2) \leqslant \max a(H_1) + \max a(H_2). \tag{1}$$

This is seen as follows; suppose $G \sim H_1 \oplus H_2$; we construct a commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
H_1 \arrow[r] \arrow[d, hook] & G_1 \arrow[r] \arrow[d] & 0 \\
H_1 \oplus H_2 \arrow[r] & G \arrow[r] \arrow[d] & 0 \\
H_2 \arrow[r] \arrow[u, hook] & G_2 \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$

with $G_1 \sim H_1$, and $G_2 \sim H_2$. The statement (1) follows, because

$$a(G) \leqslant a(G_1) + a(G_2).$$

Next we note

$$\max a(G_{n,m}) = \min (n, m). \tag{2}$$

This can be seen as follows. The Dieudonné module of $G_{n,m}$ equals

$$M = M_{n,m} = \mathscr{A} / \mathscr{A}(F^m - V^n).$$

Let $b = \min (n, m)$. Consider the A-submodule $N \subset M$ generated by

$$F^{b-1} \cdot \bar{1}, \ldots, F^{b-i} V^{i-1} \cdot \bar{1}, \ldots, V^{b-1} \cdot \bar{1}.$$

Clearly the BT group G_1, with Dieudonné module N, has $a(G_1) = b$, thus $\max a(G_{n,m}) \geqslant b$. Now suppose $n = \min (n, m)$; Let $N \subset M$ be an $\mathscr{A}$-submodule such that $Q := M/N$ has finite length. Consider the exact commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & N' \arrow[r] \arrow[d, hook, "F"'] & M' \arrow[r] \arrow[d, hook, "F"'] & Q' \arrow[r] \arrow[d, "F"] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & N/FN \arrow[r] & M/FM \arrow[r] & Q/FQ \arrow[r] & 0
\end{tikzcd}$$

of $\mathscr{A}$-modules, where $M' := M \otimes_W (W, \sigma)$ as usual. Because the length of K and of Q/FQ are equal, the snake lemma implies:

$$\dim_k (N/FN) = \dim (M/FM) = n.$$

Thus for the BT group G_1 with Dieudonné module N we have

$$a(G_1) = \dim_k (N/FN + VN) \leqslant \dim_k (N/FN) = n.$$

If $m = \min(n, m)$, consider $V: M \hookrightarrow M'$, construct a diagram analogous with the above one, and use $M'/VM' \simeq M/VM$, etc. (Alternative argument: $(G_{n,m})^t \simeq G_{m,n}$, and for any G we have $a(G) = a(G^t)$.) The facts (1) and (2) prove the first statement of the corollary.

Suppose H is given as above, and choose $F \sim H$ with $a(F) = n$ (this is possible by the first half of the proof). By Theorem (1.1) there exists an isogeny $F \to F'$ with $a(F') = 1$. We construct a sequence

$$F = F_0 \to F_1 \to \ldots \to F_t = F'$$

with $\operatorname{Ker}(F_i \to F_{i+1}) \simeq \alpha_p$, $0 \leqslant i < t$. From $F_{i+1} \simeq F_i/\alpha_p$ we obtain the exact sequence

$$0 \to \operatorname{Hom}(\alpha_p, \alpha_p) \to \operatorname{Hom}(\alpha_p, F_i) \to \operatorname{Hom}(\alpha_p, F_{i+1}),$$

hence

$$a(F_{i+1}) \geqslant a(F_i) - 1.$$

Because

$$a(F_t) = 1 \leqslant m \leqslant n = a(F_0)$$

there exists an index j such that $a(F_j) = m$, and the corollary is proved.

REMARK: The first half of the corollary was conjectured in [3], 15.8, and proved by M. POLETTI, cf. [4].

2. *Generic quotients*

Let G be a BT group over a field k; assume G is purely of local-local type or assume k is algebraically closed; let $a(G) \geqslant 1$. Let T be the tangent space to AG, thus T is an affine space of dimension a over k. By the classification of groups of height $\leqslant 1$ with the help of p-Lie algebras, cf. [1], II.7.4.3, we can identify the set of lines in $T \otimes K$ with the set of K-subgroup schemes $\alpha_p \subset G$, i.e.

$$\mathbf{P}^{a-1}(K) \cong \mathrm{Grass}\,(\mathbf{A}^1 \subset T)(K) \cong \{\alpha_p \subset G\}.$$

Let $K := k(\mathbf{P}^{a-1})$, let ξ be the generic point of $\mathbf{P}^{a-1}$, corresponding to

$$\xi : \alpha_p \to G,$$

an embedding defined over K. The field K and the quotient

$$G \to G/\xi(\alpha_p) = G_1$$

are unique up to isomorphism once G and k are given; we say that G_1 is the *generic quotient* of G. The following result can be found in private notes of D. Mumford, we present here a proof of it:

THEOREM (2.1). (Mumford). Let G be a BT group over a field k; assume k is algebraically closed. Let

$$G = G_0 \to G_1 \to G_2 \to \ldots \to G_i \to G_{i+1} \to \ldots$$

be the sequence of generic quotients. Then there exists an index $s \geqslant 0$ such that

$$a(G_s) \leqslant 1.$$

PROOF. Let $K_0 = k$, and denote by K_{i+1} an algebraic closure of $K_i(\xi_i)$, where

$$\xi_i : \alpha_p \to G_i,\ G_{i+1} = G_i/\xi_i(\alpha_p),$$

is the generic point of $\mathbf{P}^{a-1} \cong \{\alpha_p \subset G_i\}$, $a = a(G_i)$, over k. Denote by N_i the kernel of $(G \to \ldots \to G_i)$; this is a group scheme of rank p^i contained in G (if $a(G) \geqslant 1$; if $a(G) = 0$ the theorem is trivially true, so henceforth exclude this case). Let

$$\Omega := \bigcup_{i=0}^{\infty} K_i,\ Q := \bigcup_{i=0}^{\infty} N_i \subset G.$$

We choose an isogeny (over $k = K_0$)

$$\pi : \overset{<\infty}{\bigoplus} H_j \to G$$

where each of the H_j is of type $G_{n,m}$; let

$$\begin{array}{ccc} \overset{-1}{\pi} Q =: Q' & \rightarrow & \oplus H_j =: G' \\ \downarrow & & \downarrow \pi \\ Q & \longrightarrow & G. \end{array}$$

We apply Dieudonné-module theory to Q' over Ω. Note that $Q' \subset G'$, thus the Dieudonné-module $\mathscr{M}Q'$ is finitely generated, being a quotient of $\mathscr{M}G'$; thus there exists $P \subset Q'$, a subobject in Ind $(\mathscr{N}_p)$, such that P is a BT group, and Q'/P finite. Choose an Ω-isogeny

$$P' := \overset{<\infty}{\oplus} P_h \rightarrow P$$

with each P_h of type $G_{n,m}$. Consider

$$f := (P' = \oplus P_h \rightarrow P \subset Q' \subset G' = \oplus H_j);$$

from Lemma (1.2) we deduce that f is defined over a finite field k_0; thus P is defined over the same field, because $P = \mathrm{Im}\,(f) \subset G'$. We now show: there exists a field L, with $k = K_0 \subset L \subset \Omega$, with L finitely generated over k, such that $Q \subset G$ is defined over L (i.e. the subobject Q of G is already a subobject in the category Ind $(\mathscr{N}_{p,L})$); this is proved as follows: consider

$$\begin{array}{ccc} P \subset Q' & \hookrightarrow & G' \\ \downarrow & & \downarrow \\ Q'/P =: M & \hookrightarrow & G'/P; \end{array}$$

the finite group scheme M is defined over a finitely generated extension k_1 of k_0, thus Q' is defined over k_1; moreover $\pi: G' \rightarrow G$ is defined over k, and $\pi(Q') = Q$, thus $Q \subset G$ is defined over the compositum L of k_1 and k.

Now we choose an index t so that

$$k \subset L \subset K_t \subset \Omega;$$

we write $F := G/N_t = G_t$, and $F_i = G_{t+i}$, i.e. the sequence $F = F_0 \rightarrow F_1 \rightarrow \ldots$ is the sequence of generic quotients starting from F. We now show: *if* $a(F_i) > 1$ *for all* $i \geqslant 0$, *then* $AF \not\subset Q/N_t$. This can be proved as follows: choose some

$$f: \alpha_p \rightarrow F,\ f(\alpha_p) =: N \subset F,\ f \neq 0,$$

defined over L; let $\pi_i: F \rightarrow F_i$ be the quotient mapping and assume (induction hypothesis) that $\pi_i(N) \neq 0$ for some $i \geqslant 0$;

$$\begin{tikzcd} & \alpha_i \cong \pi_i(N) \arrow[r, hook] & AF_i \arrow[d, hook] & \\ \alpha_p \arrow[urr, dashed, "\xi"] \arrow[rr, "\xi"'] & & F_i \arrow[r] & F_{i+1} \end{tikzcd}$$

if $a(F_i) \geqslant 2$, the generic map $\xi : \alpha_p \to AF_i \subset F_i$ has an image which is different from $\pi_i(N)$, thus $\pi_{i+1}(N) \neq 0$. This proves $N \not\subset (N_{t+i}/N_t)$ for all i, thus $N \not\subset Q/N_t$.

Now we conclude the proof of the theorem: because Q is defined over L, and $L \subset K_t$, we know that

$$(Q/N_t) \cap AG_t$$

is defined over K_t. If $a(G_{t+i})$ would be bigger than one for all $i \geqslant 0$, then this intersection would be different from AG_t as we have seen above; because

$$\xi_t : \alpha_p \to G_t$$

is generic over K_t, this would imply

$$\xi_t(\alpha_p) \not\subset Q/N_t,$$

thus $N_{t+1} \not\subset Q = \bigcup_{j+1}^{\infty} N_j$, a contradiction. This proves there exists an index i with $a(G_{t+i}) = 1$, and the theorem is proved.

COROLLARY (2.2). Let $Q \subset G$ be as above; then Q equals the local-local part of G.

PROOF. Let M be a local-local finite subgroup scheme of G; suppose $M' \subset M$ and $M' \neq M$ implies $M' \subset Q$ (i.e. we apply induction on ${}^p\log$ rank M). Choose $M' \subset M$ such that $M/M' \cong \alpha_p$, and take an index u so that $M' \subset N_u$; apply the theorem to the case $F := G/N_u = G_u$,

$$M \to M/M' \to G/N_u =: F = F_0 \to F_1 \to \ldots \to F_v \to F_{v+1};$$

then there exists an index v such that $a(F_v) = 1$, thus the composite map

$$\alpha_p = M/M' \to F_v \to F_{v+1}$$

is zero, because $AF_v = \xi(\alpha_p) = \mathrm{Ker}\,(F_v \to F_{v+1})$; thus $M \subset Q$, and the corollary is proved.

From the theorem we deduce another proof of Theorem (1.1): we assume $k = \bar{k}$ and F purely of local-local type; we write $F = F_0$, $k = k_0$, and we construct inductively schemes S_i, BT groups $F_i \to S_i$, morphisms $S_{i+1} \to S_i$ and quotients

$$\begin{tikzcd} F_i \times S_{i+1} \arrow[rr] \arrow[dr] & & F_{i+1} \arrow[dl] \\ & S_{i+1} & \end{tikzcd}$$

for $i=0, 1, \ldots,$ as follows: we assume inductively that $a(F_{i,s})$ is constant for all geometric points s of S_i. Then $AF_i \to S_i$ exists, it is a finite flat group scheme over S_i, which locally is isomorphic with $(\alpha_p)^a$, $a=a(F_{i,s})$. Then we construct S'_{i+1} as the $\mathbf{P}^{a-1}$-bundle over S_i defined by the vector bundle of tangent vectors to AF_i; over S'_{i+1} we have the BT group $F_i \times S'_{i+1}$, and a finite flat group scheme $N \subset F_i \times S'_{i+1}$, which locally is isomorphic with α_p. The number $a(F_{i,s}/N_s)$ is upper semi-continuous, thus contstant on a non-empty open set $S_{i+1} \subset S'_{i+1}$; we define $F_{i+1}=(F_i/N)|S_{i+1}$. Clearly these schemes S_i are irreducible, the generic fibre of $F_i \to S_i$ is exactly the i-th generic quotient of $F=F_0$; by Theorem (2.1) there exists an index i such that $a(F_{i,s})=1$. The scheme S_i is of finite type over k, thus there exists a k-point $s \in S_i(k)$, and $F' := F_{i,s}$ is the BT group we are looking for, which yields a second proof of Theorem (1.1).

Mathematisch Instituut
Universiteit van Amsterdam
Roetersstraat 15, Amsterdam

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